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Statistical approximation by a modification of q-Meyer-König and Zeller operators

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ABSTRACT

In this work, we introduce a modification of the q-Meyer-König and Zeller operators, and investigate the Korovkin type statistical approximation properties of this modification via A-statistical convergence. Also we prove that this modification provides a better estimation than the q-MKZ operators on the interval $[\alpha_n, 1) \subset [\frac{1}{2}, 1)$ by means of the modulus of continuity.

1. Introduction

In order to give the monotonicity properties, a modification of the classical Meyer-König and Zeller (MKZ) operators in [1] is defined by Cheney and Sharma [2] as follows:

$$M_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n+1,k}(x), \quad f \in C[0,1) \text{ and } x \in [0,1), \tag{1.1}$$

where

$$m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n.$$

Notice that these operators are also known as Bernstein power series in the literature.

Let $e_i(x)$ be the monomials defined as $e_i(x) = x^i \ (i = 0, 1, 2)$.

It is obvious that the operators $M_n(f;x)$ preserve the monomials, i.e., $M_n(e_0;x) = e_0(x)$ and $M_n(e_1;x) = e_1(x)$, for all $n \in \mathbb{N}$. It is also known that

$$e_2(x) \le M_n(e_2;x) \le e_2(x) + \frac{x}{n}.$$

So the second central moment of the operators $M_n(f;x)$ satisfies the following inequality:

$$M_n((t-x)^2;x) \le \frac{e_1(x)}{n} = \frac{x}{n}.$$

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In a recent paper, Özarslan and Duman [3] introduced a different modification of the classical MKZ operators (1.1), and proved that their modification provides a better estimation than the operators (1.1) on the interval $[\frac{1}{2}, 1)$. In this paper we introduce a modification of the q-MKZ operators, and investigate the Korovkin type approximation properties of this modification via A-statistical convergence. We also compute a rate of A-statistical convergence of these q-type operators by means of the modulus continuity, and we prove that this modification provides a better estimation than the q-MKZ operators on the interval $[\alpha_n, 1) \subset [\frac{1}{2}, 1)$.

2. Construction of the operators

In this part we will construct a modification of the MKZ operators based on q-integers.

Before proceeding further we recall some definitions concerning q-integers. For any non-negative integer r, the q-integer of the number r is defined by

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q} & \text{if } q \neq 1 \\ r & \text{if } q = 1 \end{cases}$$

where q is a positive real number. The q-factorial is defined as

$$[r]_q! = \begin{cases} [1]_q[2]_q \dots [r]_q & \text{if } r = 1, 2, \dots \\ 1 & \text{if } r = 0 \end{cases}$$

and the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

It is obvious that q-binomial coefficients reduce to the ordinary case when $q = 1$. Details of q-integers can be found in [4] and [5].

First, we recall the following q-MKZ operators introduced by Doğru and Duman in [6]:

$$M_n(f; q; x) = u_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{q^n [k]_q}{[n+k]_q}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k \tag{2.1}$$

where

$$f \in C[0, a], a \in (0, 1), q \in (0, 1], n \in \mathbb{N}$$

and

$$u_{n,q}(x) = \prod_{s=0}^{n} (1 - xq^s).$$

Now, we consider the following operators:

$$R_n(f; q; x) = u_{n,q}(x) \sum_{k=0}^{\infty} f\left(q^n \sqrt{\frac{[k]_q [k-1]_q}{[n+k]_q [n+k-1]_q}}\right) \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k \tag{2.2}$$

where

$$f \in C[0, a],\ a \in (0, 1),\ q \in (0, 1],\ n \in \mathbb{N}$$

and

$$u_{n,q}(x) = \prod_{s=0}^{n} (1 - xq^s).$$

It is clear that the operators $R_n(f; q; x)$ are positive and linear. A few calculations reveal that

$$R_n(e_0; q; x) = e_0(x) \quad \text{and} \quad R_n(e_2; q; x) = q^{2n} e_2(x). \tag{2.3}$$

To obtain Korovkin type statistical approximation properties of our modification, we require the following lemma.

Lemma 2.1. *For all $n \in \mathbb{N}$ and $x \in [0, 1)$, $q \in (0, 1]$ we have*

$$xq^{n-1} - q^{n-1} \frac{(1 - xq^n)}{[n]_q} \leq R_n(e_1; q; x) \leq xq^{n+1} + \frac{(1 - xq^n)\, xq^n}{[n]_q}, \tag{2.4}$$

where $e_1(t) = t$.

Proof. Using the facts that

$$[k-1]_q \le [k]_q\,,$$
$$\sqrt{\frac{[k]_q\,[k-1]_q}{[n+k]_q\,[n+k-1]_q}} \le \frac{[k]_q}{[n+k-1]_q},$$
$$[n+k]_q = q\,[n+k-1]_q + 1,$$

we have

$$\begin{aligned} R_n(t;q;x) &= u_{n,q}(x)\sum_{k=0}^{\infty} q^n \sqrt{\frac{[k]_q\,[k-1]_q}{[n+k]_q\,[n+k-1]_q}}\,\frac{[n+k]_q!}{[n]_q!\,[k]_q!}x^k \\ &\le u_{n,q}(x)\sum_{k=1}^{\infty} q^n \frac{[k]_q}{[n+k-1]_q}\,\frac{[n+k]_q!}{[n]_q!\,[k]_q!}x^k \\ &= u_{n,q}(x)\sum_{k=1}^{\infty} q^n \frac{q\,[n+k-1]_q+1}{[n+k-1]_q}\,\frac{[n+k-1]_q!}{[n]_q!\,[k-1]_q!}x^k \\ &= u_{n,q}(x)\sum_{k=1}^{\infty} q^{n+1} \frac{[n+k-1]_q!}{[n]_q!\,[k-1]_q!}x^{k-1}x + u_{n,q}(x)\sum_{k=1}^{\infty} q^n \frac{1}{[n+k-1]_q}\,\frac{[n+k-1]_q!}{[n]_q!\,[k-1]_q!}x^k \\ &= xq^{n+1} + \frac{(1-xq^n)\,xq^n}{[n]_q}u_{n-1,q}(x)\sum_{k=1}^{\infty}\frac{[n+k-2]_q!}{[n-1]_q!\,[k-1]_q!}x^{k-1} \\ &= xq^{n+1} + \frac{(1-xq^n)\,xq^n}{[n]_q}u_{n-1,q}(x)\sum_{k=0}^{\infty}\frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!}x^{k} \\ &= xq^{n+1} + \frac{(1-xq^n)\,xq^n}{[n]_q}. \end{aligned}$$

Then we can write

$$R_n(e_1;q;x) \le xq^{n+1} + \frac{(1-xq^n)\,xq^n}{[n]_q}. \tag{2.5}$$

On the other hand, since

$$\sqrt{\frac{[k]_q\,[k-1]_q}{[n+k]_q\,[n+k-1]_q}} > \frac{[k-1]_q}{[n+k]_q}$$
$$[k-1]_q = \frac{[k]_q-1}{q},$$

we obtain

$$\begin{aligned} R_n(t;q;x) &= u_{n,q}(x)\sum_{k=0}^{\infty} q^n \sqrt{\frac{[k]_q\,[k-1]_q}{[n+k]_q\,[n+k-1]_q}}\,\frac{[n+k]_q!}{[n]_q!\,[k]_q!}x^k \\ &\ge u_{n,q}(x)\sum_{k=0}^{\infty} q^n \frac{[k-1]_q}{[n+k]_q}\,\frac{[n+k]_q!}{[n]_q!\,[k]_q!}x^k \\ &= u_{n,q}(x)\sum_{k=0}^{\infty} q^n \frac{[k]_q-1}{q\,[n+k]_q}\,\frac{[n+k]_q!}{[n]_q!\,[k]_q!}x^k \\ &= u_{n,q}(x)\sum_{k=1}^{\infty} q^{n-1} \frac{[n+k-1]_q!}{[n]_q!\,[k-1]_q!}x^k - \frac{(1-xq^n)}{[n]_q}u_{n-1,q}(x)\sum_{k=0}^{\infty} q^{n-1}\frac{[n+k-1]_q!}{[n-1]_q!\,[k]_q!}x^k \\ &= xq^{n-1} - q^{n-1}\frac{(1-xq^n)}{[n]_q}. \end{aligned}$$

Thus we can write

$$R_n(e_1;q;x) \ge xq^{n-1} - q^{n-1}\frac{(1-xq^n)}{[n]_q}. \tag{2.6}$$

Using (2.5) and (2.6), the proof is completed. □

At this point let us recall the following A-statistical approximation theorem given by Gadjiev and Orhan [7].

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix. If the sequence of positive linear operators L_n from $C[a, b]$ into $C[a, b]$ satisfies the conditions*

$$st_A - \lim_n \|L_n(e_i; x) - e_i\| = 0 \quad \text{with } e_i(t) = t^i \text{ for } i = 0, 1, 2$$

then, for all $f \in C[a, b]$, we have

$$st_A - \lim_n \|L_n(f; x) - f\| = 0.$$

Now let $A = (a_{jn})$ be a non-negative regular summability matrix. Then replace q in (2.2) by a sequence (q_n) in the interval $(0, 1]$ so that

$$st_A - \lim_n q_n^n = 1 \quad \text{and} \quad st_A - \lim_n \frac{1}{[n]_{q_n}} = 0. \tag{2.7}$$

Indeed, we can construct a sequence (q_n) satisfying (2.7). For example, take

$$a_{jn} = \begin{cases} 1, & \text{if } n = j \text{ and } j \neq m^2 \ (m = 1, 2, 3 \ldots) \\ \frac{1}{2}, & \text{if } j = m^2 \text{ and } n = j \text{ or } n = (m-1)^2 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $A = (a_{jn})$ is a regular and triangular matrix. We define the sequence (q_n) by

$$q_n = \begin{cases} 0, & \text{if } n = m^2 \ (m = 1, 2, 3 \ldots) \\ \dfrac{n}{n+1}, & \text{if } n \neq m^2. \end{cases}$$

We can say that $st_A - \lim_n q_n^n = 1$ but observe that (q_n^n) is non-convergent in the ordinary sense. On the other hand, if $n \neq m^2$, then

$$\frac{1}{[n]_{q_n}} = \frac{1 - \frac{n}{n+1}}{1 - \left(\frac{n}{n+1}\right)^n} = \frac{\frac{1}{n+1}}{1 - \left(1 - \frac{1}{n+1}\right)^n}.$$

This guarantees that $st_A - \lim_n \frac{1}{[n]_{q_n}} = 0$.

We are now ready to obtain the following result for the operators (2.2).

Theorem 2.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let (q_n) be a sequence satisfying (2.7). Then, for all $f \in C[0, a]$, $0 < a < 1$, we have*

$$st_A - \lim_n \|R_n(f; q_n; .) - f\| = 0.$$

Proof. By (2.3), it is clear that

$$st_A - \lim_n \|R_n(e_0; q_n; .) - e_0\| = 0, \tag{2.8}$$

and

$$\|R_n(e_2; q_n; x) - e_2\| \leq 1 - q_n^{2n}. \tag{2.9}$$

For a given $\varepsilon > 0$, we define the following sets:

$$U := \{n : \|R_n(e_2; q_n; .) - e_2\| \geq \varepsilon\} \quad \text{and} \quad U' := \left\{n : 1 - q_n^{2n} \geq \varepsilon\right\}.$$

From (2.9) we can see that $U \subseteq U'$ and then, for each $j \in \mathbb{N}$, that

$$0 \leq \sum_{n \in U} a_{jn} \leq \sum_{n \in U'} a_{jn}. \tag{2.10}$$

Letting $j \to \infty$ in (2.10) and using (2.7) we conclude that

$$\lim_j \sum_{n \in U} a_{jn} = 0,$$

which gives

$$st_A - \lim_n \|R_n(e_2; q_n; .) - e_2\| = 0. \tag{2.11}$$

Finally, by (2.6), we get

$$\|R_n(e_1; q_n; .) - e_1\| \le 1 - q_n^{n-1} + \frac{q_n^{n-1}}{[n]_{q_n}}. \tag{2.12}$$

Since, for all $n \in \mathbb{N}$, $0 < q_n^{n-1} \le q_n \le 1$, one can get $st_A - \lim_n q_n^n = 1$. So, by (2.7) we observe that

$$st_A - \lim_n (1 - q_n^{n-1}) = st_A - \lim_n \frac{q_n^{n-1}}{[n]_{q_n}} = 0.$$

Now define the following sets:

$$D := \{n : \|R_n(e_1; q_n; .) - e_1\| \ge \varepsilon\},$$

$$D_1 := \left\{n : 1 - q_n^{n-1} \ge \frac{\varepsilon}{2}\right\} \quad \text{and} \quad D_2 := \left\{n : \frac{q_n^{n-1}}{[n]_{q_n}} \ge \frac{\varepsilon}{2}\right\}.$$

Then we obtain from (2.12) that $D \subseteq D_1 \cup D_2$. Hence we have, for all $j \in \mathbb{N}$, that

$$0 \le \sum_{n \in D} a_{jn} \le \sum_{n \in D_1} a_{jn} + \sum_{n \in D_2} a_{jn}.$$

Taking the limit $j \to \infty$, we immediately get that

$$st_A - \lim_n \|R_n(e_1; q_n; .) - e_1\| = 0. \tag{2.13}$$

Now using (2.8), (2.11) and (2.13), the proof is completed. □

3. Rates of *A*-statistical convergence

In this section, we compute the rate of A-statistical convergence of the operators $R_n(f; q; x)$ given by (2.2) to $f(x)$ by means of the modulus of continuity. Also, we show that this modification provides a better estimation than the q-MKZ operators on the interval $[\alpha_n, 1) \subset [\frac{1}{2}, 1)$.

Let $f \in C[0, 1)$. The modulus of continuity of f, denoted as $w(f, \delta)$, is defined to be

$$w(f, \delta) = \sup_{\substack{|y-x| \le \delta \\ x, y[0,1)}} |f(y) - f(x)|.$$

Then it is known that $\lim_{\delta \to 0} w(f, \delta) = 0$ for $f \in C[0, 1)$; and also, for any $\delta > 0$ and each $x, y \in [0, 1)$ we have

$$|f(y) - f(x)| \le w(f, \delta)\left(\frac{|y - x|}{\delta} + 1\right). \tag{3.1}$$

Now using this terminology we first have the following.

Theorem 3.1. *Let (q_n) be sequence such that $0 < q_n \le 1$ for each n. Then for every $f \in C[0, a]$, $0 < a < 1$, and $x \in [0, a]$, we have*

$$|R_n(f; q_n; x) - f(x)| \le 2w(f, \delta_n^*), \tag{3.2}$$

where

$$\delta_n^* = \sqrt{x^2(1 - 2q_n^{n-1} + q_n^{2n}) + 2xq_n^{n-1}\frac{(1 - xq_n^n)}{[n]_{q_n}}}. \tag{3.3}$$

Proof. Let $f \in C[0, a]$. By linearity and positivity of the operators $R_n(f; q_n; x)$ we get, for all $n \in \mathbb{N}$ and $x \in [0, a]$, that

$$|R_n(f; q_n; x) - f(x)| \le R_n(|f(t) - f(x)|; q_n; x). \tag{3.4}$$

Now using (3.1) in inequality (3.4) we have, for any $\delta > 0$, that

$$|R_n(f; q_n; x) - f(x)| \le w(f, \delta)\left\{1 + \frac{1}{\delta} R_n(|t - x|; q_n; x)\right\}. \tag{3.5}$$

Applying the Cauchy–Schwarz inequality for positive linear operators it follows from (3.5) that

$$
\begin{aligned}
|R_n(f;q_n;x)-f(x)| &\le w(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{R_n((t-x)^2;q_n;x)}\right\}\\
&= w(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{R_n(e_2;q_n;x)-2xR_n(e_1;q_n;x)+x^2R_n(e_0;q_n;x)}\right\}.
\end{aligned}
$$

Using (2.3) in (2.4) in the last inequality, we can write

$$
|R_n(f;q_n;x)-f(x)| \le w(f,\delta)\left\{1+\frac{1}{\delta}\sqrt{x^2-2x\left(xq_n^{n-1}-q_n^{n-1}\frac{\left(1-xq_n^n\right)}{[n]_{q_n}}\right)+x^2}\right\}. \tag{3.6}
$$

Choosing $\delta := \delta_n^*$ as in (3.3) it follows from (3.6) that the proof is completed. □

If (q_n) satisfies (2.7), then the sequence $\left(\delta_n^*\right)$ is A-statistically null, which yields that $st_A-\lim_n w(f,\delta_n^*)=0$. So, Theorem 3 shows the rate of A-statistical convergence of the operators $R_n(f;q;x)$ to $f(x)$.

For the q-MKZ operators given in (2.1), it is known that (see [6])

$$
|M_n(f;q_n;x)-f(x)| \le 2w(f,\delta_n) \quad \text{for } f\in C[0,a],\ 0<a<1 \text{ and } x\in[0,a],
$$

where

$$
\delta_n = \sqrt{x^2(1-2q_n^n+q_n^{2n+1})+\frac{xq_n^{2n}}{[n]_{q_n}}}. \tag{3.7}
$$

Now we claim that our rate of convergence (3.2) is better than the estimate (3.7) on the interval $[\alpha_n,1)\subset[\frac{1}{2},1)$ for

$$
\alpha_n = \frac{1-\frac{q_n^{n+1}}{2}}{1-\frac{q_n^{n+1}}{2}+\frac{q_n^{2n+1}}{2}}.
$$

Indeed,

$$
\begin{aligned}
(\delta_n)^2 &= x^2(1-2q_n^n+q_n^{2n+1})+\frac{xq_n^{2n}}{[n]_{q_n}}\\
&= x^2(1-2q_n^{n-1}+q_n^{2n})+(2q_n^{n-1}-2q_n^n)x^2+(q_n^{2n+1}-q_n^{2n})x^2+\frac{xq_n^{2n}}{[n]_{q_n}}\\
&= x^2(1-2q_n^{n-1}+q_n^{2n})-\frac{2q_n^{n-1}q_n^nx^2}{[n]_{q_n}}+\frac{2q_n^{n-1}x^2}{[n]_{q_n}}-\frac{q_n^{2n}x^2}{[n]_{q_n}}+\frac{q_n^{3n}x^2}{[n]_{q_n}}+\frac{xq_n^{2n}}{[n]_{q_n}}.
\end{aligned}
$$

Thus from (3.3), we have

$$
\begin{aligned}
(\delta_n)^2-\left(\delta_n^*\right)^2 &= \frac{2q_n^{n-1}x^2}{[n]_{q_n}}-\frac{q_n^{2n}x^2}{[n]_{q_n}}+\frac{q_n^{3n}x^2}{[n]_{q_n}}+\frac{xq_n^{2n}}{[n]_{q_n}}-\frac{2q_n^{n-1}x}{[n]_{q_n}}\\
&= \frac{1}{[n]_{q_n}}\left\{2q_n^{n-1}x\left(\frac{q_n^{n+1}}{2}-1\right)+2q_n^{n-1}x^2\left(1-\frac{q_n^{n+1}}{2}+\frac{q_n^{2n+1}}{2}\right)\right\}\\
&= \frac{1}{[n]_{q_n}}\left\{2q_n^{n-1}x\left(x\left(1-\frac{q_n^{n+1}}{2}+\frac{q_n^{2n+1}}{2}\right)-\left(1-\frac{q_n^{n+1}}{2}\right)\right)\right\}\\
&\ge 0
\end{aligned}
$$

for $x\in[\alpha_n,1)$ and it is obvious that $\frac{1}{2}\le\alpha_n<1$ which corrects our claim.

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